Q1
If \( f(x) \) is defined for \( x \geq 0 \), then the Laplace transform of \( f(x) \) is given by:

\[
L \{ f(x) \} = \int_0^\infty e^{-sx} f(x) \, dx
\]

1) Prove that \( L \{ x^n \} = \frac{n}{s} L \{ x^{n-1} \} \).

2) Find:
   \[
   \text{i) } L \{ 4x^2 - 5\sin 3t \} \quad \text{ii) } L \left\{ (1+e^{2t})^2 \right\} \quad \text{iii) } L^{-1} \left\{ \frac{(s+2)^2}{s^3} \right\}
   \]

3) Use the Laplace transform to solve:
   \[
y' - y = 1 \quad ; \quad y(0) = 0
   \]

Q2
The unit step function is defined to be:

\[
u(t-a) = \begin{cases} 
0 & ; \ 0 \leq t < a \\
1 & ; \ t \geq a
\end{cases}
\]

1) Given:

\[
f(t) = \begin{cases} 
2 & ; \ 0 \leq t < 3 \\
-2 & ; \ t \geq 3
\end{cases}
\]

   i) Write \( f(t) \) in terms of the unit step functions.
   ii) By using (i) or any other method, find the Laplace transform of \( f(t) \).

2) If \( a > 0 \) prove that

\[
L \{ f(t-a)u(t-a) \} = e^{-as} L \{ f(t) \}
\]

3) Find

\[
L \{ e^{-2t} \cos 4t \}
\]

Q3
The convolution of \( f(t) \) and \( g(t) \) is defined as

\[
f * g = \int_0^t f(\tau)g(t-\tau) \, d\tau
\]

1) How we can find Laplace transform of the convolution of two functions?
2) Is the convolution of two functions commutative?
3) Evaluate

\[
\int_0^t e^\tau \sin(t-\tau) \, d\tau
\]
Two functions $f_1$ and $f_2$ are said to be orthogonal on interval $[a,b]$ if the inner product of them equals zero.

1) Determine if the given functions are orthogonal on $[0,2]$ or not

$$f_1(x) = e^x, \quad f_2(x) = xe^{-x} - e^{-x}$$

2) Show that the set $\{1, \cos x, \cos 2x, \ldots\}$ is orthogonal on the interval $[-\pi, \pi]$.

3) Expand

$$f(x) = \begin{cases} 0 & ; \quad -\pi < t < 0 \\ 1 & ; \quad 0 \leq t < \pi \end{cases}$$

in a Fourier series.